CURVES IN $\mathbb{R}^3$ AND FRENET APPARATUS

Note Title 10/1/2008

Let I be an open interval in $\mathbb{R}$

$a < t < b$ or even $a < t$ (infinite open interval)

or even $t < b$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

$\alpha_1, \alpha_2, \alpha_3 : \mathbb{R} \to \mathbb{R}$ (real valued fns.)

$\alpha_1, \alpha_2, \alpha_3$: EUCLIDEAN COORDINATE FNS.

If α_1, α_2 and α_3 are differentiable then α is differentiable.

DEFINITION: CURVE in $\mathbb{R}^3$ is a <u>DIFFERENTIABLE</u> function $\alpha : I \to \mathbb{R}^3$ from an open interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^3$

EXAMPLES :

1. STRAIGHT LINE :

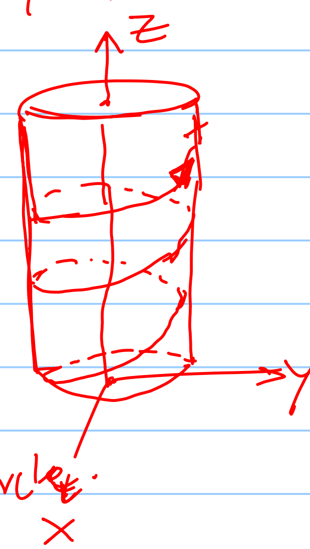
$$\alpha(t) = \mathbb{p} + tq \quad , \quad q \neq 0$$
$$= (p_1 + tq_1 \,,\, p_2 + tq_2 \,,\, p_3 + tq_3)$$

is a straight line through the point $\mathbb{p} = \alpha(0)$ in the direction q.

2. HELIX

$$\alpha : t \to (a\cos t,\ a\sin t,\ 0)$$
is a circle on the XY plane

If every point on the circle is displaced linearly in the Z axis then it is an Helix.



$$\alpha : t \to (a\cos t,\ a\sin t,\ bt)$$

where $a > 0$, $\underbrace{b \neq 0}$

if $b = 0$, it is a circle.

a: radius of the cylinder

TANGENT VECTOR

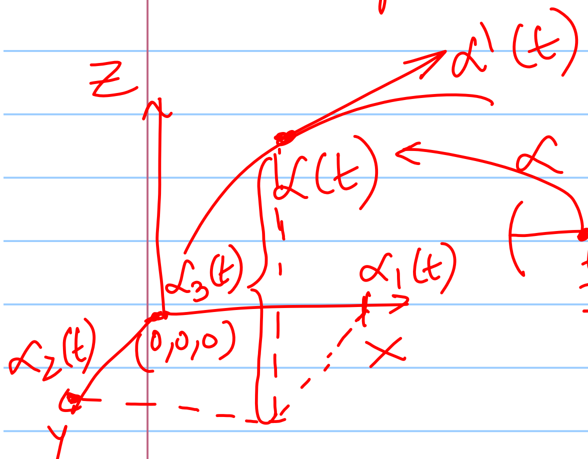
Let $\alpha : I \to \mathbb{R}^3$ be the curve with

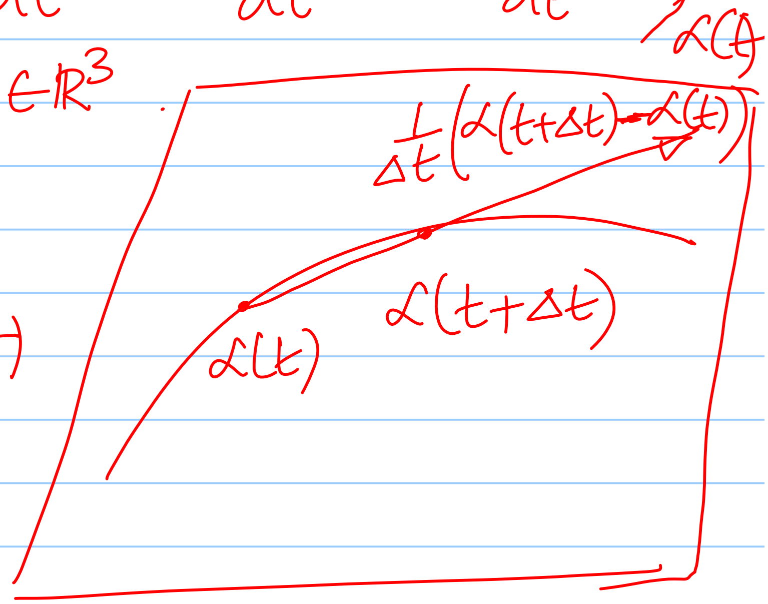
$\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

For each $t \in I$, the velocity vector of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t) \right)$$

at the point $\alpha(t) \in \mathbb{R}^3$.





EXAMPLE.

STRAIGHT LINE

$$\alpha(t) = (p_1 + tq_1 , \ p_2 + tq_2 , \ p_3 + tq_3)$$

$$\alpha(t+\Delta t) = (p_1 + (t+\Delta t)q_1 , \ p_2 + (t+\Delta t)q_2 , \ p_3 + (t+\Delta t)q_3)$$

$$\alpha(t+\Delta t) - \alpha(t) = (\Delta t q_1 , \Delta t q_2 , \Delta t q_3)$$

$$\frac{1}{\Delta t}(\alpha(t+\Delta t) - \alpha(t)) = (q_1, q_2, q_3) = q$$

the direction vector

ALL velocity vectors are parallel to each other and is same as the direction vector. Only the point of application changes as t changes.

HELIX $\quad \alpha(t) = (a\cos t, \ a\sin t , \ bt)$

$$\alpha'(t) = (-a\sin t , \ a\cos t , b)_{\alpha(t)}$$

HELIX raises constantly since z coordinate of $\alpha'(t)$ is a constant

REPARAMETRIZATION OF CURVES

Given any curve a new curve can be constructed that follows the same path as the given curve.

These two curves which look exactly the same would be different in their velocity vectors $\alpha'(t)$

Constructing such new curves is called reparametrization

DEFINITION. Let $\alpha : I \to \mathbb{R}^3$ be a curve
Let $h : J \to I$ be a differentiable function from an open interval $J \subseteq \mathbb{R}$ to I.

Then $\beta = \alpha(h) : J \to \mathbb{R}^3$ is a curve called reparametrization of α by h.

For each $s \in J$, the new curve β is at the point $\beta(s) = \alpha(h(s))$

Check the velocity of β:

$$\beta'(s) = \alpha'(h(s)) \cdot h'(s)$$

EXAMPLE

$$\begin{tikzcd} s \in J \subseteq \mathbb{R} \arrow[r, "h"] \arrow[rr, bend left, "\beta"] & t \in I \subseteq \mathbb{R} \arrow[r, "\alpha"] & \mathbb{R}^3 \end{tikzcd} \qquad \beta(s) = \alpha(h(s))$$

Let $\alpha(t) = \left(\sqrt{t},\ t \cdot \sqrt{t},\ 1-t\right)$ on

$$I: 0 < t < 4$$

Let $h(s) = s^2$ on $J: 0 < s < 2$

then

$$\beta(s) = \alpha(h(s)) = \alpha(s^2)$$

$$= \left(s,\ s^3,\ 1-s^2\right)$$

$$\alpha'(t) = \left(\frac{1}{2\sqrt{t}},\ \frac{3}{2}\sqrt{t},\ -1\right)$$

$$\beta'(s) = \left(1,\ 3s^2,\ -2s\right)$$

$$\beta'(s) = \alpha'(h(s)) \cdot h'(s)$$

$$\alpha'(h(s)) = \left(\frac{1}{2\sqrt{h(s)}},\ \frac{3}{2}\sqrt{h(s)},\ -1\right)$$

$$= \left(\frac{1}{2\sqrt{s^2}},\ \frac{3}{2}\sqrt{s^2},\ -1\right) = \left(\frac{1}{2s},\ \frac{3}{2}s,\ -1\right)$$

$\boxed{h'(s) = 2s}$ $\therefore$ $\beta'(s) = \left(1,\ 3s^2,\ -2s\right)$

LENGTH OF A CURVE SEGMENT

Curve segment of a curve α. is a part of α for $[a,b] \subset I$, a closed interval.

Speed of a curve at a point $\alpha(t)$: $\|\alpha'(t)\|$

Lenth of a curve segment $\displaystyle\int_a^b \|\alpha(t)\|\, dt$.

Regular Curve: A curve where $\|\alpha'(t)\| \neq 0$ for any value of $t \in I$

Reparametrization of a regular curve to Unit speed curve: ARC-LENGTH PARAMETRIZATION

Let us reparameterize the curve α starting from a point $\alpha(a)$, $a \in I$

Let $\displaystyle s(t) = \int_a^t \|\alpha'(t)\| \cdot dt$

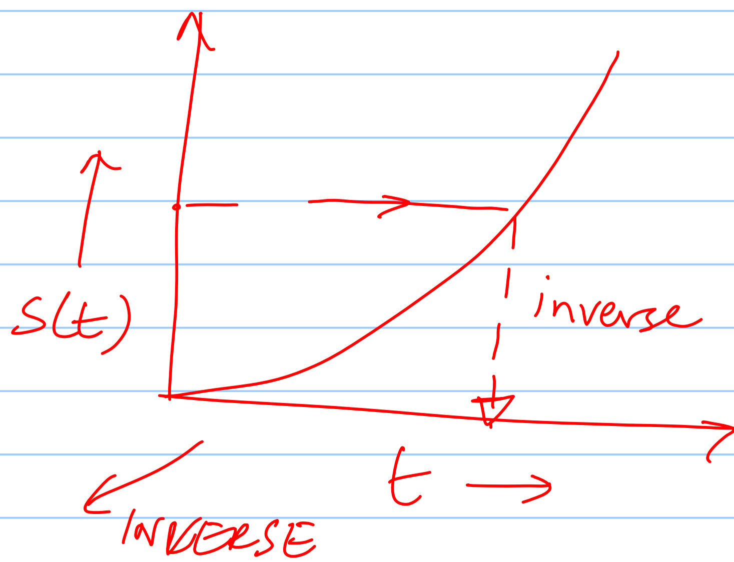
Thus $\dfrac{ds}{dt} = \|\alpha'\|$. Since α is regular $\|\alpha'\| \neq 0$

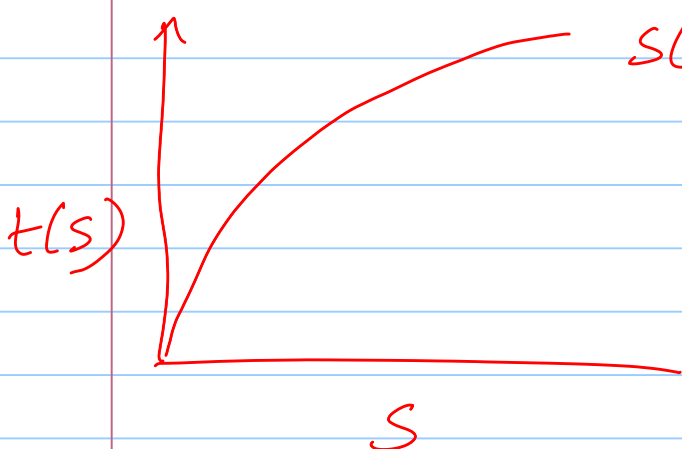
Therefore $\frac{ds}{dt} > 0$

$\therefore$ function's have a maxima or minima

$\therefore$ s is monotonic.

$\therefore$ s^{-1}





$\frac{dt}{ds}$ is also > 0 since $\frac{ds}{dt} > 0$

$\beta(s) = \alpha(t(s))$ is the reparametrization we need.

$$\beta'(s) = \alpha'(t(s)) \cdot \frac{dt}{ds}(s)$$

$$\|\beta'(s)\| = \frac{dt}{ds}(s) \cdot \|\alpha'(t(s))\| = \frac{dt}{ds}(s) \cdot \frac{ds}{dt}(t(s)) = 1$$

EXAMPLE ARC LENGTH PARAMETRIZATION

HELIX

$$\alpha(t) = (a\cos t,\ a\sin t,\ bt)$$

$$\alpha'(t) = (-a\sin t,\ a\cos t,\ b)$$

$$\|\alpha'(t)\|^2 = \alpha'(t)\cdot\alpha'(t)$$

$$= a^2\sin^2 t + a^2\cos^2 t + b^2$$

$$= a^2 + b^2$$

$$\|\alpha'(t)\| = \sqrt{a^2+b^2} = c \quad (\text{a constant})$$

$$s(t) = \int_0^t c\cdot dt = ct.$$

arc length

$$s = ct \quad \therefore\ t = \frac{s}{c}$$

$$\therefore\ \beta(s) = \alpha\left(\frac{s}{c}\right) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{bs}{c}\right)$$

It is easy to see that $\|\beta'(s)\| = 1$

What is the derivative of a unit vector field?
(like the tangents of β, since $\|\beta'(s)\|=1$)

Let $\|V\|=1 \Rightarrow \|V\|^2=1 \Rightarrow V\cdot V=1$

Differentiating both sides of the eqn.

$$V'\cdot V + V\cdot V' = 0$$

$$2V'\cdot V = 0$$

$$V'\cdot V = 0$$

$\therefore$ V and V' are $\perp r$ to each other.

$\therefore$ For an arc-length parametrized curve β, since $\|\beta'\| = \|T\| = 1$

Tangent vector of β

T' is $\perp r$ to T.

That is the normal vector at the point of β. Let $\|T'\| = \kappa$ (curvature)

$$N = \frac{T'}{\|T'\|} = \frac{T'}{\kappa}$$

FRENET FORMULAS (Frenet Apparatus)

- Finding an orthogonal coordinate system at every point on the curve β.
- There are many such systems. One of which is Frenet frame
- They should change smoothly over the curve.

Let $\beta : I \rightarrow \mathbb{R}^3$ be a unit-speed curve.

$\|\beta'(s)\| = 1$ for each $s \in I$

Let $T = \beta'$ be the unit tangent vector on β

$T' = \beta''$ measures the way curve is turning in $\mathbb{R}^3$

(It is a straight line if $T' = \beta'' = 0 \Rightarrow$ No turning)

$\kappa(s) = \|T'(s)\|$ is the curvature at $\beta(s)$

$$\vec{N}(s) = \frac{T'(s)}{\|T'(s)\|} = \frac{T'(s)}{\kappa}. \qquad \boxed{N \cdot T = 0}$$

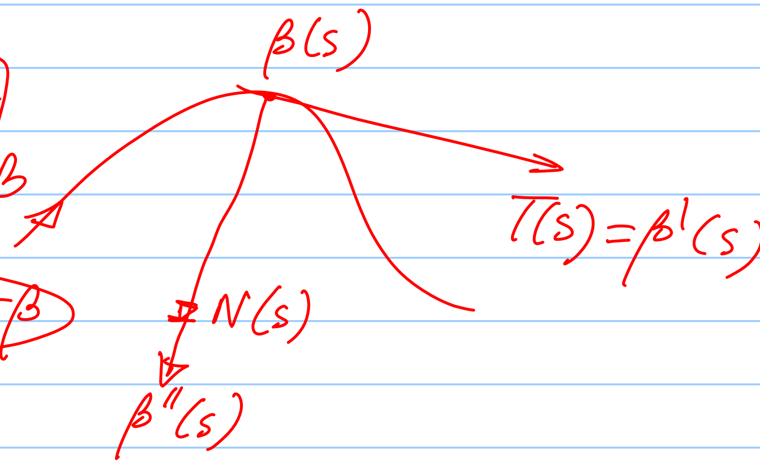
Unit vector

Let $\kappa \neq 0$ (In order to compute N)

$\therefore \kappa > 0$

$N = \dfrac{T'}{\kappa}$ tells the direction in which β is turning at each pt.

$\boxed{\text{N is called the PRINCIPAL NORMAL VECTOR FIELD of } \beta}$



The vector field $B = T \times N$ on β is called the <u>BINORMAL</u> vector field of β.

T, N, and B are mutually orthogonal to each other and form an orthogonal coordinate frame at every point on the curve. This frame is called a <u>Frenet frame</u>

The idea is to use the Frenet frame to study the curves as much as possible instead of using the natural frame field ($U_1, U_2, U_3 - xyz$) since the Frenet field has all the information about the curve whereas natural frame has none.

Derivatives of T, N, and B

$T' = \kappa N$

We know $B' \cdot B = 0$ since B is a unit vector field.

$B \cdot T = 0 \Rightarrow B' \cdot T + B \cdot T' = 0$

$\Rightarrow B' \cdot T + \underbrace{B \cdot \kappa \cdot N}_{=0} = 0$

$\Rightarrow B' \cdot T = 0$

$\Rightarrow B'$ and T are $\perp$r to each other

B' is $\perp$r to Both B and T, $\therefore B'$ is $\parallel$ to N

$\Rightarrow \underline{B' = -\tau N}$ $\quad \tau$ is the TORSION

T, N and B form an orthogonal coordinate system.

$\therefore$ any vector V can be expressed as

$$V = (V \cdot T)\, T + (V \cdot B)\, B + (V \cdot N)\, N$$

Let us now express N' in this form.

$$N' = (N' \cdot T) \cdot T + (N' \cdot B) \cdot B + \underbrace{(N' \cdot N)}_{=0} \cdot N$$

$\therefore$ N is a unit vector field.

$N \cdot T = 0$

$\Rightarrow N' \cdot T + N \cdot T' = 0$

$N' \cdot T + N \cdot \kappa \cdot N = 0$

$\boxed{N' \cdot T = -\kappa}$

$N \cdot B = 0$

$\Rightarrow N' \cdot B + N \cdot B' = 0$

$N' \cdot B + N \cdot -\tau \cdot N = 0$

$N' \cdot B = \tau$

$$\therefore \quad N' = -\kappa T + \tau B$$

$$\boxed{\begin{aligned} T' &= \quad\;\; \kappa N \\ N' &= -\kappa T \qquad + \tau B \\ B' &= \quad -\tau N \end{aligned}}$$

EXAMPLE Frenet frame for a unit-speed helix

$$\beta(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{bs}{c}\right)$$

where $c = \sqrt{a^2+b^2}$ and $a > 0$

$$T(s) = \left(-\frac{a}{c}\cdot\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right)$$

$$T'(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c},\ -\frac{a}{c^2}\sin\frac{s}{c},\ 0\right)$$

$$\kappa(s) = \|T'(s)\| = \frac{a}{c^2} = \frac{a}{a^2+b^2} > 0$$

$$N(s) = \frac{T'(s)}{\kappa(s)} = \left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right)$$

[N points straight into the axis of the cylinder of the helix]

$$B(s) = \left(\frac{b}{c}\sin\frac{s}{c},\ -\frac{b}{c}\cos\frac{s}{c},\ \frac{a}{c}\right)$$

$$B'(s) = \left(\frac{b}{c^2}\cos\frac{s}{c},\ \frac{b}{c^2}\sin\frac{s}{c},\ 0\right) \quad \therefore \boxed{\tau = \frac{b}{a^2+b^2}}$$

FRENET APPROXIMATION OF A CURVE

Let us do a Taylor series expansion of β for small values of s.

$$\beta(s) \sim \beta(0) + s.\beta'(0) + \frac{s^2}{2}\beta''(0) + \frac{s^3}{6}\beta'''(0)$$

$$\beta'(0) = T_0 \qquad (T_0 = T(0))$$

$$\beta''(0) = \kappa_0 N_0$$

$$\beta''' = (\kappa N)' = \kappa' \cdot N + \kappa \cdot N'$$

$$= \kappa' \cdot N + \kappa \cdot (-\kappa T + \tau B)$$

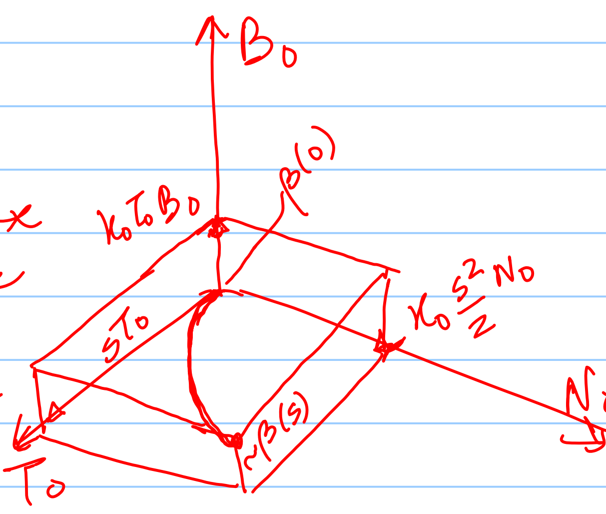
$$\beta(s) \sim \beta(0) + sT_0 + \frac{s^2}{2} \cdot \kappa_0 N_0 + \frac{s^3}{6} \cdot \kappa_0 \tau_0 \cdot B_0$$

One term: Same as starting pt.

Two terms: linear approx. along the tgt. vector

Three terms: Best quadratic approx. Parabola on T-N plane (osculating plane)

Four terms: Twist from the osculating plane - Torsion.



Note when $\kappa_0 = 0$, it is a straight line approximation.

$\therefore$ A curve is a st. line iff $\|T'\| = \kappa = 0$

Note when $\tau = 0$, the curve is planar.

Let β be a unit speed curve with $\kappa > 0$

β is a plane curve $\iff \tau = 0$

β is a unit speed curve with a CONSTANT CURVATURE $\kappa > 0$ and TORSION $\tau = 0$

the β is a part of a circle

(Circle has const. curvature (> 0) & Torsion $= 0$ since it is planar)