Vector field

Note Title 4/16/2006

Def: A vector field V on $\mathbb{R}^3$ is a function that assigns to each point p of $\mathbb{R}^3$ a tangent vector $V(p)$ to $\mathbb{R}^3$ at p.

Roughly speaking, a vector field is just a big collection of arrows, one at each point of $\mathbb{R}^3$.

$$(V+W)(p) = V(p) + W(p)$$

$$(fV)(p) = f(p)\, V(p)$$

Def Let $U_1 = (1,0,0)$, $U_2 = (0,1,0)$ and $U_3 = (0,0,1)$ be the vector fields on $\mathbb{R}^3$. for each pt. p of $\mathbb{R}^3$. These three vector fields collectively called the NATURAL FRAME FIELD on $\mathbb{R}^3$

lemma If V is a vector field on $\mathbb{R}^3$ there are three uniquely determined real-valued functions v_1, v_2, v_3 on $\mathbb{R}^3$ such that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3.$$

v_1, v_2, and v_3 are called EUCLIDEAN COORDINATE fns. of V

Def. Let f be a differentiable real-valued function on $\mathbb{R}^3$, and let v_p be a tangent vector to $\mathbb{R}^3$. Then the number

$$v_p[f] = \frac{d}{dt}\left(f(p+tv)\right)\Big|_{t=0}$$

is called the <u>derivative of f wrt v_p</u>

If v_p is a unit vector then $v_p[f]$ is called the directional derivative of f wrt v_p.

Lemma $v_p = (v_1, v_2, v_3)$ is a tangent vector to $\mathbb{R}^3$

then $$v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)$$

Derivative of f wrt natural frame is:

$F = \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}$. This is a vector and its component along v_p is its directional derivative. Hence $v_p[f] = F \cdot v = \sum v_i \frac{\partial f}{\partial x_i}(p)$

Theorem

1. $(aV_p + bW_p)[f] = aV_p[f] + bW_p[f]$

2. $V_p[af + bg] = aV_p[f] + bV_p[g]$

3. $V_p[fg] = V_p[f] \cdot g(p) + f(p) \cdot V_p[g]$

Directional Derivative along a Vector field is analogous.

COVARIANT DERIVATIVE

Def. Let W be a vector field on $\mathbb{R}^3$ and let v be a tangent vector field to $\mathbb{R}^3$ at the point p. Then the covariant derivative of W wrt v is the tangent vector

$$\nabla_v W = W(p+tv)'(0)$$

at the point p.

How to compute Covariant Derivatives?

Example:

$$W = x^2 U_1 + yz\, U_3$$

$v = (-1, 0, 2) \qquad p = (2, 1, 0)$

$$p + tv = (2-t, 1, 2t)$$

$$W(p+tv) = (2-t)^2 U_1 + 2t\, U_3$$

$$\nabla_v W = W(p+tv)'(0) = -4U_1(p) + 2U_3(p)$$

ALTERNATE WAY OF COMPUTING COVARIANT DERIVATIVES

$$W = w_1 U_1 + w_2 U_2 + w_3 U_3$$

w_1, w_2, and w_3 are scalar coordinate fns.

Compute the rate of change of these individual functions in the direction of v.

$V[w_1]$, $V[w_2]$ and $V[w_3]$

The covariant derivative of the vector field W wrt V is.

$$\nabla_V W = V[w_1]U_1 + V[w_2]U_2 + V[w_3]U_3$$

<u>Previous Example</u> $W = x^2 U_1 + yz U_3$
$V = (-1, 0, 2)$ $p = (2, 1, 0)$

$w_1 = x^2$ $w_2 = 0$ $w_3 = yz$

$$w_1' = \left(\frac{\partial w_1}{\partial x}, \frac{\partial w_1}{\partial y}, \frac{\partial w_1}{\partial z} \right) = (2x, 0, 0)$$

$$V[w_1] = V \cdot w' = -2x$$

Similarly $V[w_2] = 0$ $V[w_3] = 2y$

$$\nabla_V W = -2x U_1 + 2y U_3$$

$$(\nabla_V W)(p) = -4 U_1(p) + 2 U_3(p)$$

Another Example

$W = x U_1 + x^2 U_2 - z^2 U_3$

$V = (1, -1, 2) \qquad p = (1, 3, -1)$

$w_1' = (1, 0, 0) \qquad w_2' = (2x, 0, 0)$

$w_3' = (0, 0, -2z)$

$V[w_1] = 1 \qquad V[w_2] = 2x \quad V[w_3] = -4z$

$\nabla_V W = (1, 2x, -4z)$

$(\nabla_V W)(p) = U_1(p) + 2U_2(p) + 4U_3(p)$