Surfaces in $\mathbb{R}^3$

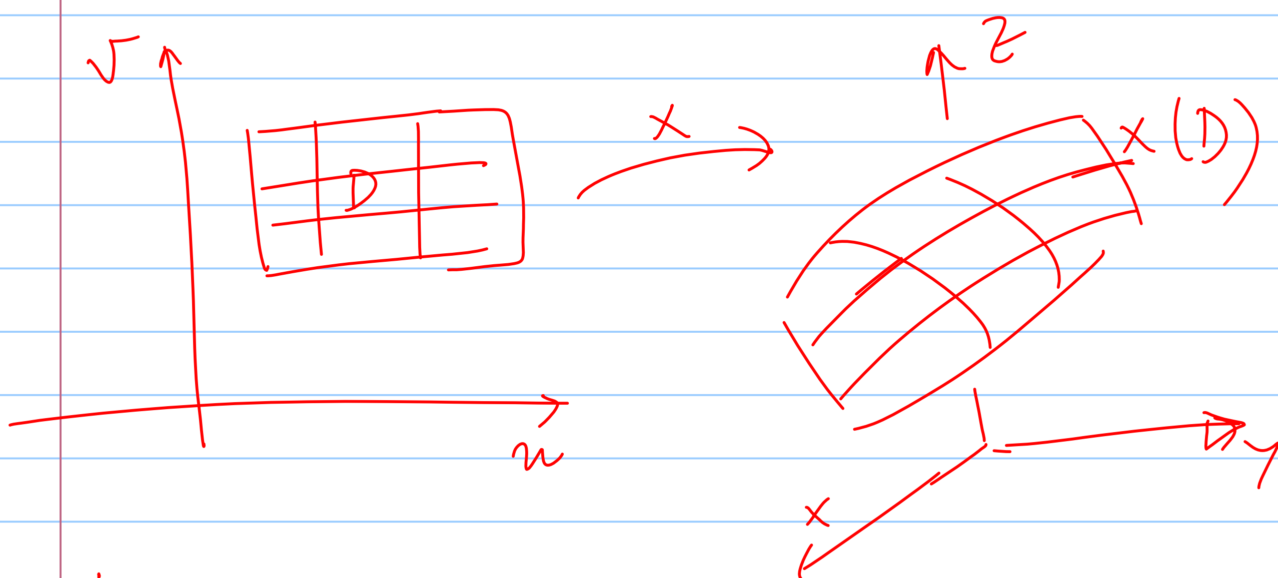
Note Title 10/6/2008

Definition. A coordinate patch

$x : D \to \mathbb{R}^3$ is a one-to-one regular mapping of an open set $D \subseteq \mathbb{R}^2$ into $\mathbb{R}^3$

(Regular - Derivative exists everywhere)

D is also called the domain of parametrization



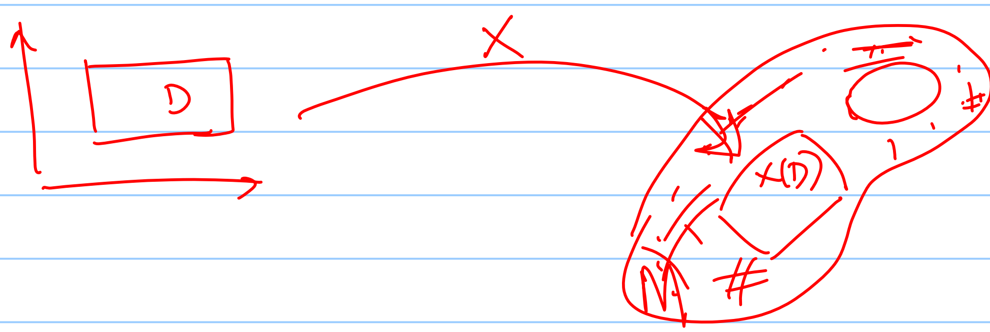
proper patch $x^{-1} : x(D) \to D$ is continuous

Image of a patch x is contained in Surface M

$\to$ x is a patch of M.

Definition: A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point p of M there exists a proper patch in M whose image contains a neighborhood of p in M

x can cover parts of M.
Many patches together may cover entire M.



MONGE PATCH Let f be any differentiable function on an open set D in $\mathbb{R}^2$. Then the function $x : D \to \mathbb{R}^3$ s.t. $x(u,v) = (u, v, f(u,v))$ is a proper patch called MONGE PATCH.

IMPLICIT SURFACE

Let g be a differentiable real valued function on $\mathbb{R}^3$ and c any number. The subset

$$M : g(x,y,z) = c \text{ of } \mathbb{R}^3 \text{ is a surface}$$

if the differential dg is not zero at any point of M.

This is a theorem called the implicit function theorem and the surface M is called the implicit surface.

Proof If p is a point on M, we must find a proper patch covering a neighborhood of p in M

$$dg = \frac{\partial g}{\partial x} \cdot dx + \frac{\partial g}{\partial y} \cdot dy + \frac{\partial g}{\partial z} \cdot dz$$

and $dg \neq 0$. $\therefore$ atleast one, say $\left(\frac{\partial g}{\partial z}\right)(p) \neq 0$

In this case, the Implicit fn. theorem says that, near p, $g(x,y,z) = c$ can be solved for z.

More precisely, it asserts that

there is a differentiable real-valued function h defined on a neighborhood D of (p_1, p_2) such that

(1) For each (u,v) in D, $(u,v,h(u,v))$ lies in M. That is $g(u,v,h(u,v)) = c$

(2) Points of the form $(u,v,h(u,v))$ with $(u,v) \in D$ fill the neighborhood of p.

(all $(u,v,h(u,v))$ are in M, and all points in M in the neighborhood of p are of the form $(u,v,h(u,v))$)

Hence neighborhood of M is a Monge patch.

$$X(u,v) : (u, v, h(u,v)).$$

Since p is an arbitrary point in M, every point in M is covered by a Monge patch.

Hence M is a surface.

PARAMETRIZATION

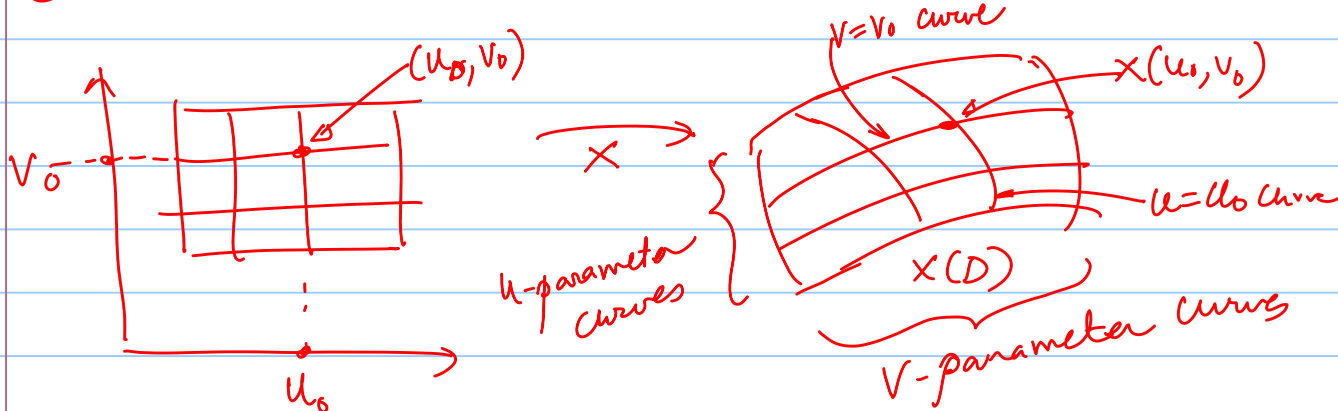
A regular mapping $X : D \to R^3$ whose image lies in a surface M is called a parametrization of the region $X(D)$ of M.

Curves on Surfaces

Let $x : D \to \mathbb{R}^3$ be a coordinate patch.

Holding u or v constant in a function
$(u,v) \to x(u,v)$ produces curves

For each point (u_0, v_0), two such curves pass through it, $u \to (u, v_0)$ and $v \to (u_0, v)$
called the u- and v- parameter curves



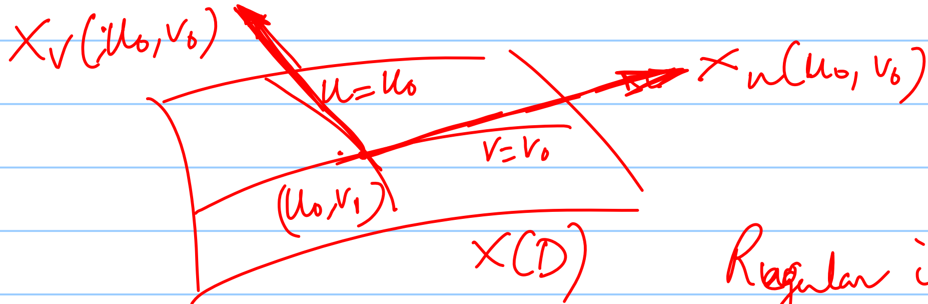
Definition $\quad x : D \to \mathbb{R}^3$ is a patch.
For each pt. $(u_0, v_0) \in D$

(1) The velocity vector at u_0 of the u-parameter curve, $v = v_0$ is $x_u(u_0, v_0)$

(2) ... at v_0 of the v-parameter curve, $u = u_0$ is $x_v(u_0, v_0)$.

The vectors x_u and x_v are called partial velocities of x at (u_0, v_0)

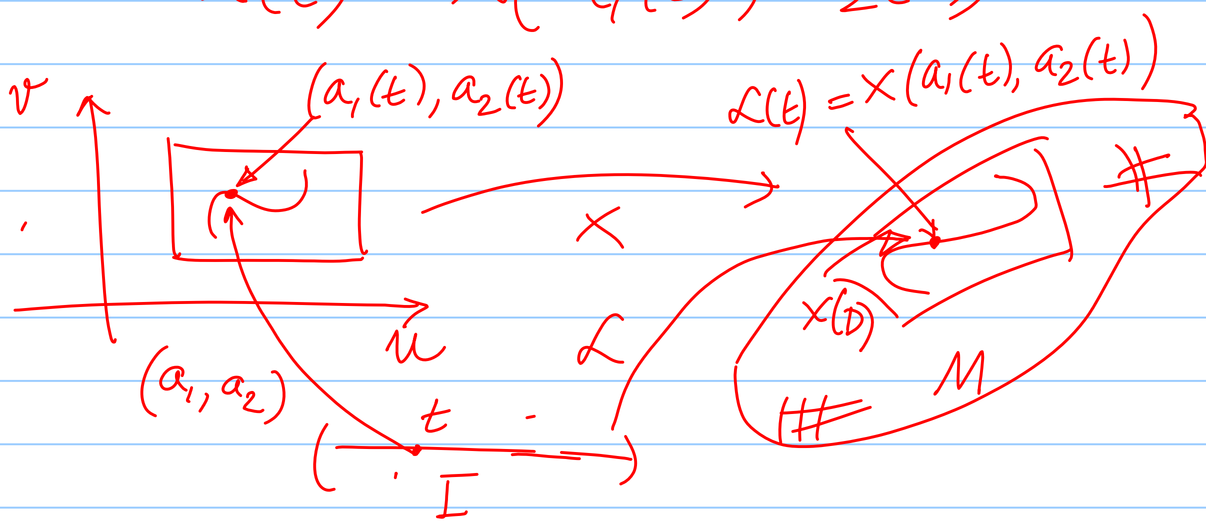
Normal vector is $X_u \times X_v$



Regular if $X_u \times X_v \neq 0$

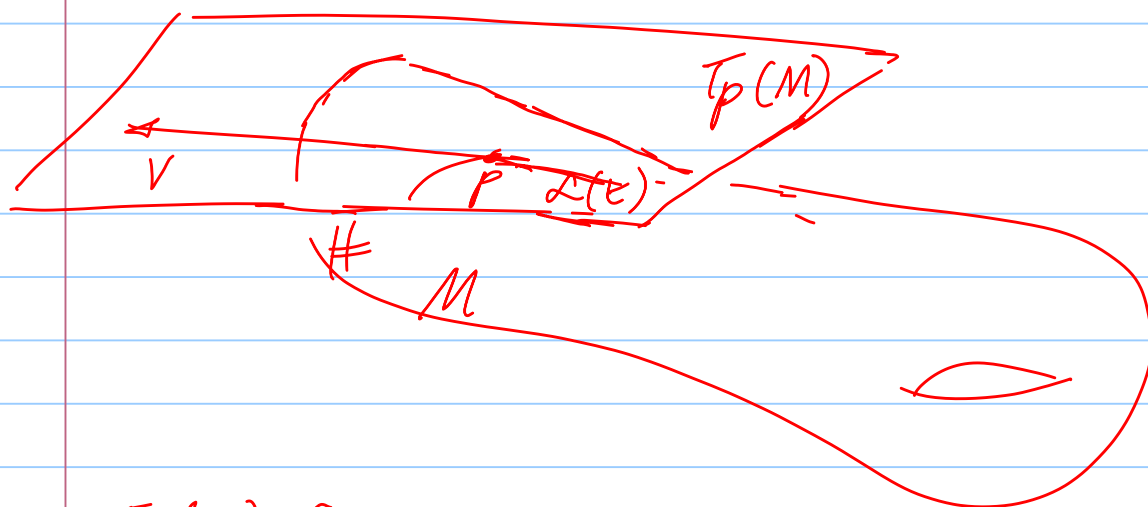
Generic curves $\alpha : I \to M$

$$\alpha(t) = X(a_1(t), a_2(t))$$



Definition Let $p \in M \subset \mathbb{R}^3$. A tgt. vector v to $\mathbb{R}^3$ at p is tangent to M at p provided v is a velocity of some curve in M.

Def. The set of all tgt. vectors to M at p is called the tangent plane of M at p, and is denoted by $T_p(M)$



$T_p(M)$ is a 2D-subspace of $T_p(\mathbb{R}^3)$

$T_p(M)$ is the plane formed by X_u and X_v at p.

Normal vector of M at p is $\perp$r to $T_p(M)$

Lemma If $M: g = c$ is a surface in $\mathbb{R}^3$, then the gradient vector field $\nabla g = \sum \left(\frac{\partial g}{\partial x_i}\right) U_i$ (considered only at points of M) is a non-vanishing normal vector field on the entire surface M.