Shape Operators

Note Title 4/17/2006

Def. A Euclidean vector field Z on a surface M in $\mathbb{R}^3$ is a function that assigns to each point p of M a tangent vector $Z(p)$ to $\mathbb{R}^3$ at p

[It is basically a vector field with three components]

Def. A Euclidean vector field V for which each vector . p) is tangent to M is called a tangent vector field on M

Def. A Euclidean vector field U on M is a normal vector field on M if each vector $U(p)$ is normal to M

Def. Let p be a pt. on M, and let v be a tangent vector to M at p. Then the <u>shape operator $S_p(v)$</u> is defined as <u>$S_p(v) = -\nabla_v U$</u> where U is a unit normal vector field on a neighborhood of p in M.

<u>Lemma</u> Shape operator is a linear operator

$$S_p : T_p(M) \to T_p(M).$$

Let v be a <u>tangent vector to R^3</u>

Y & Z be vector fields on R^3

$$v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$$

Similarly $v[U \cdot U] = 2 \nabla_v U \cdot U(p)$

$$= \underline{-2 S_p(v) \cdot U(p)} = 0 \quad [\text{since } U \cdot U = 1]$$

$\boxed{S_p(v) \perp U}$

Linearity is a consequence of a linearity property of the covariant Derivatives.

$$S_p(av+bw) = -\nabla_{av+bw} U$$
$$= -(a\nabla_v U + b\nabla_w U)$$
$$= aS_p(V) + bS_p(W)$$

EXAMPLES

1. Sphere radius r, v is a tangent vector. The normal vector U topples forward in the exact same direction of v itself. Therefore $S(v) = -cv$

$$U = \frac{1}{r}\sum x_i U_i$$

$$\nabla_v U = \frac{1}{r}\sum v[x_i]U_i(p) = \frac{v}{r}$$

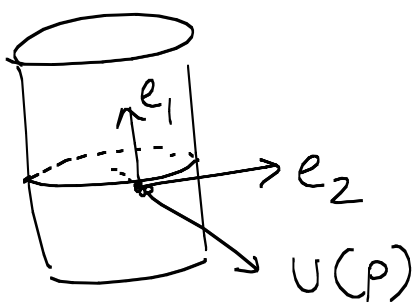
$$S(v) = -\frac{v}{r}$$

2. Plane P in $\mathbb{R}^3$

A normal vector field U on P is parallel in $\mathbb{R}^3$.

Hence $S(v) = -\nabla_v U = 0$.

3. Cylinder



$S_p(e_1) = 0$

$S_p(e_2) = -\frac{e_2}{r}$

4. Saddle surface $z = xy$.

x and y-axes are tangent to the surface. When we move along $u_1 = (1,0,0)$ the normal vector remains $\perp r$ to u_1 and rotates. Hence $\nabla_{u_1} U = -u_2$. Similarly $\nabla_{u_2} U = -u_1$. For any other direction

$$S(au_1 + bu_2) = bu_1 + au_2$$

In case of curve, there is only one direction to move and κ and τ measure the rate of change of the T and B (and so N).

For surface only one unit vector field is intrinsically determined - normal vector field.

At each pt. there is a whole plane of directions in which U can move. Hence numbers like κ and τ are insufficient and it requires a linear operator S.

Shape operator is a Symmetric Linear Operator

$$S : T_pM \longrightarrow T_pM.$$

$$S(V) \cdot W = S(W) \cdot V$$