NORMAL CURVATURE

Note Title 4/18/2006

We saw that covariant derivative

$$\nabla_v W = W(p+tv)'(0)$$

i.e. Rate of change of W in the direction of v

But if W is defined only along v, then clearly, $\nabla_v W = W'(0)$

[We have earlier seen T, N, B etc on curves in R^3. These vectors are defined only on the points on the curve, and their derivatives N' and B' are actually $\nabla_T N$ and $\nabla_T B$]

Assume that the Normal vector field U of the surface is defined (restricted) to a curve α on M.

α' is the tangent vector (at p)

$\nabla_{\alpha'} U = U'$ (as in the case of N' and B')

Hence $\boxed{S_p(\alpha') = -U'}$

Proof that U' is also in the tangent plane of M at p.

$U \perp \alpha' (= T)$

$\therefore \quad U = aN + bB$

$U' = aN' + bB'$

$= a(-\kappa T + \tau B) + b(-\tau N)$

$= -a\kappa T + a\tau B - b\tau N$

$U' \cdot U = (a\tau \cdot b)\, B \cdot B - (b\tau \cdot a)\, N \cdot N$

$= ab\tau - ab\tau = 0$

Lemma If α is a curve in $M \subset \mathbb{R}^3$, then

$$\alpha'' \cdot U = S(\alpha') \cdot \alpha'$$

Proof $\alpha' \cdot U = 0 \quad (\because \alpha' = T)$

$$\alpha'' \cdot U + \alpha' \cdot U' = 0$$

$$\alpha'' \cdot U = -\alpha' \cdot U' = S(\alpha') \cdot \alpha'$$

The acceleration component (α'') normal to the surface (U) is dependent on the velocity (α') and the shape operator.

Def: Let u be a unit tangent vector to M at p. Then the number

$\boxed{k(u) = S(u) \cdot u}$ is called the

<u>normal curvature of M</u> in the direction of u

— $k(u) = S(u) \cdot u = S(-u) \cdot (-u) = k(-u)$

— $k(u) = S(u) \cdot u = \alpha''(0) \cdot U(p)$

$$= \kappa(0) \cdot \cos\vartheta$$

$\kappa(0)$ is the curvature of the unit speed curve with tangent vector u at p $(=\alpha(0))$ and ϑ is the angle between the principal normal $N(0)$ and the surface normal $U(p)$.

There is a natural way to avoid ϑ (make it 0 or π).

- Given u (the tangent vector) and U (unit surface normal) construct P, plane cutting M at p; a curve σ called the NORMAL

SECTION of M in the u direction.
Assume σ is a unit speed curve.

$$\sigma(0) = p$$

$$\sigma'(0) = u$$

$$\sigma''(0) = \kappa_\sigma(0) \cdot N(0)$$

and $N(0)$ is on plane $\mathbb{P}$ since σ is a planar curve.

$$\therefore \quad N(0) \cdot U(p) = \pm 1$$

$$K(u) = \sigma''(0) \cdot U(p)$$

$$= \kappa_\sigma(0) \cdot N(0) \cdot U(p)$$

$$= \pm \kappa_\sigma(0)$$

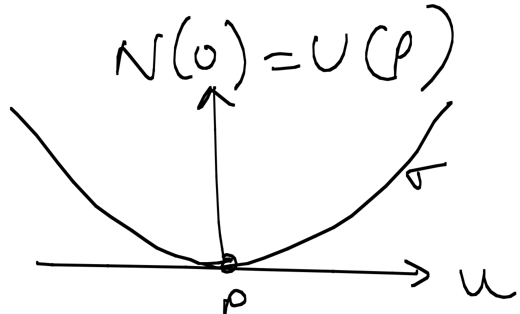
NORMAL CURVATURE IN THE 'U DIRECTION IS SAME AS THE PRINCIPAL CURVATURE OF THE NORMAL SECTION CURVE IN THE U-DIRECTION UPTO SIGN

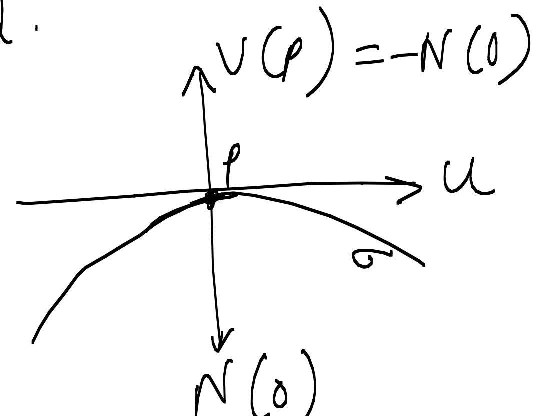
SIGN OF THE NORMAL CURVATURE

① $\kappa(u) > 0$; $N(0) = U(p)$
Normal section σ is bending toward $U(p)$. Thus in the u direction the surface M is bending toward $U(p)$

② $\kappa(u) < 0$ then $N(0) = -U(p)$
Normal section σ is bending away from $U(p)$. Thus in the u-direction the surface M is bending away from $U(p)$.

③ $\kappa(u) = 0$ $\kappa_\sigma(0) = 0$ and $N(0)$ is undefined.





PRINCIPAL CURVATURES

The maximum and minimum values of the normal curvature $k(u)$ of M at p are called the principal curvatures of M at p and are denoted by k_1 and k_2.

The directions in which these extreme values occur are called principal directions of M at p.

Unit vectors in these directions are called principal vectors of M at p.

Def A pt. p is umbilic if the normal curvature $k(u)$ is constant on all unit tangent vectors u at p.

MAIN THEOREM

1) If p is an umbilic of M the the shape operator S at p is just a scalar multiplication by $k = k_1 = k_2$.

2) If p is non-umbilic $k_1 \neq k_2$, then there are exactly two principal directions and these are orthogonal.

3) Furthermore if e_1 and e_2 are principal vectors in these directions then $S(e_1) = k_1 e_1$, $S(e_2) = k_2 (e_2)$

Hence the <u>principal curvatures</u> of M at p are the <u>eigenvalues</u> of S and the <u>principal vectors</u> are the <u>eigenvectors</u> of S.

PROOF

Let k take its maximum value k_1 at e_1.

So. $k_1 = S(e_1) \cdot e_1$

Let $e_2 \perp e_1$ (need not be a principal vector)

Any unit tangent vector

$$u = u(\theta) = c e_1 + s e_2$$

$(c = \cos\theta$ and $s = \sin\theta)$

$$k(\theta) = k(u(\theta))$$

$$= S(c e_1 + s e_2) \cdot (c e_1 + s e_2)$$

$$= c^2 S(e_1) \cdot e_1 + cs\, S(e_1) \cdot e_2$$

$$+ cs\, S(e_2) \cdot e_1 + s^2 S(e_2) \cdot e_2$$

$$k(\theta) = c^2 S_{11} + 2cs\, S_{12} + s^2 S_{22}$$

$$\frac{dk}{d\theta}(\theta) = 2sc(S_{22} - S_{11}) + 2(c^2 - s^2) S_{12}$$

If $\nu = 0$, $u(0) = e_1$ $\quad c = 1, s = 0$

and $\kappa(\nu) = k_1$ which is the maximum value. Hence $\frac{d\kappa}{d\nu}(0) = 0$

$$2sc(S_{22} - S_{11}) + 2(c^2 - s^2) S_{12} = 0$$

$c = 1,\ s = 0$

$$\Rightarrow \quad 2(1) S_{12} = 0$$

$$\Rightarrow \quad S_{12} = 0$$

$$\Rightarrow \quad S(e_1) \cdot e_2 = 0$$

$$\Rightarrow \quad S(e_1) \perp e_2$$

$$\Rightarrow \quad S(e_1) \parallel e_1$$

Since $k_1 = \kappa(e_1) = S(e_1) \cdot e_1$

$$\boxed{S(e_1) = k_1 e_1}$$

Similarly when $\nu = \frac{\pi}{2}$, $c = 0$, $s = 1$,

$u = e_2$; $S_{21} = S_{12} = 0$; $S(e_2) \perp e_1$

$S(e_2) \parallel e_2$

$$K(\nu) = K\left(\frac{\pi}{2}\right) = c^2 S_{11} + 2sc\, S_{12} + s^2 S_{22}$$

$$= S_{22}$$

$$= S(e_2) \cdot e_2$$

Hence $S(e_2) = S_{22} e_2$

Rewriting the equation for $K(\nu)$

$$K(\nu) = c^2 K_1 + s^2 S_{22}$$

$K(\nu)$ is an convex combination of K_1 and S_{22}. The extremal values of $K(\nu)$ are K_1 and S_{22}. Since K_1 is the maximum value, S_{22} has to be the minimum value.

These extremal values are taken by k only when $c = \pm 1, s = 0$ or $c = 0, s = \pm 1$.

$\therefore \quad S(e_1) = k_1 e_1$ and $S(e_2) = k_2 (e_2)$

When p is umbilic

$$k = k(e_1) = k(e_2) = S_{11} = S_{22}$$

$$k(v) = c^2 S_{11} + 2sc S_{12} + s^2 S_{22}$$

$$= c^2 k + s^2 k \quad (\because S_{12} = 0)$$

$$= k \quad (\because c^2 + s^2 = 1)$$

$$S(e_1) = k_1 e_1 \quad S(e_2) = k_2 e_2$$

$\therefore$ k_1 and k_2 are eigenvalues and e_1 and e_2 are principal vectors.

IMPORTANT COROLLARY

k_1, k_2 : principal curvatures

e_1, e_2 : principal vector

Let $u = \cos\theta \, e_1 + \sin\theta \, e_2$

$$\boxed{k(u) = k_1 \cos^2\theta + k_2 \sin^2\theta}$$

QUADRATIC APPROXIMATION OF A SURFACE

Let ① p is at the origin.

② the tangent plane T_pM is the x-y plane

③ x and y axes are principal directions

① + ② $\Rightarrow$ $f^0 = 0$ $f_x^0 = 0$, $f_y^0 = 0$

Taylor series expansion

$$f(x,y) = f^0 + f_x^0 x + f_y^0 y$$
$$+ \frac{1}{2}\left(f_{xx}^0 x^2 + 2 f_{xy}^0 xy + f_{yy}^0 y^2\right)$$
$$+ \text{ higher order terms.}$$

$$f(x,y) \approx \frac{1}{2}\left(f_{xx}^0 x^2 + 2 f_{xy}^0 xy + f_{yy}^0 y^2\right)$$

[Refer to the notes "MORE ON SHAPE OPERATORS"]

$u_1 = (1,0,0) \quad u_2 = (0,1,0)$ at $p=0$

$$S(u_1) = -\nabla_{u_1} U = f_{xx}^0 u_1 + f_{xy}^0 u_2$$

$$S(u_2) = -\nabla_{u_2} U = f_{xy}^0 u_1 + f_{yy}^0 u_2$$

Since u_1 and u_2 are principal vectors (condition 3),

$$k_1 = f_{xx}^0, \quad k_2 = f_{yy}^0, \quad f_{xy}^0 = 0$$

Substituting

$$\therefore \; f(x,y) \simeq \frac{1}{2}(k_1 x^2 + k_2 y^2)$$

The shape of M near p is approximately the same as that of the surface

$$\boxed{M' : z = \frac{1}{2}(k_1 x^2 + k_2 y^2)}$$

near 0. M' is called the Quadratic approx of M near p.

Gaussian Curvature

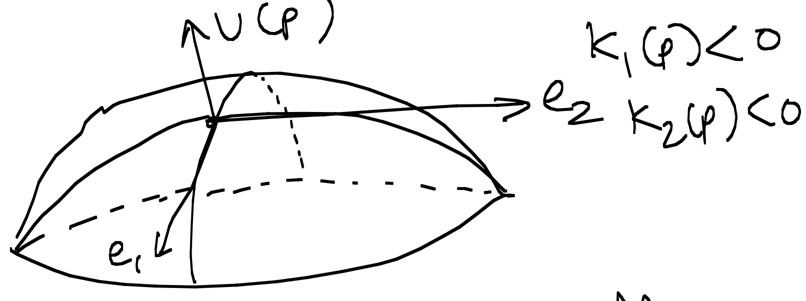
$$K = k_1 k_2 = \det(S)$$

Mean Curvature

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}\operatorname{trace}(S)$$

$\boxed{K > 0} \Rightarrow k_1, k_2 > 0$ or
$k_1, k_2 < 0$

M is bending away from its tangent plane T_pM in all directions at p.



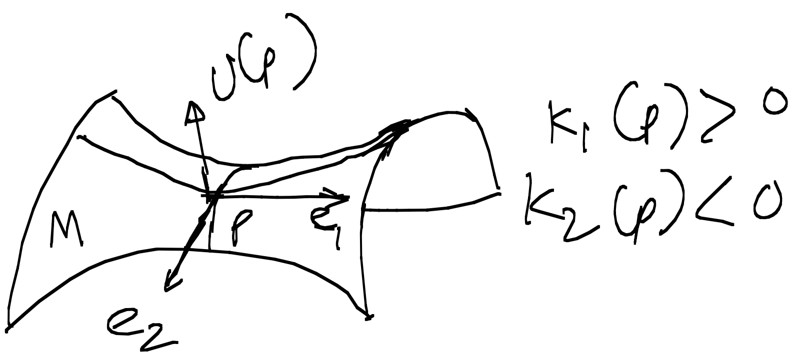
Quadratic approx of M near p is a paraboloid.

$$2z = k_1(p)\, x^2 + k_2(p)\, y^2$$

$\boxed{K(p) < 0}$ $\quad k_1(p) > 0, \quad k_2(p) < 0$

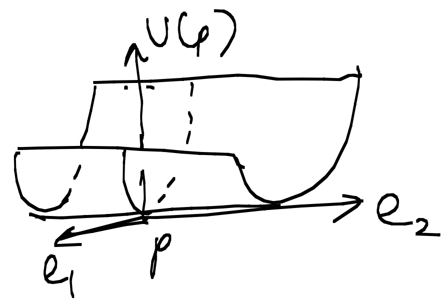
M is saddle shaped near p.
The quadratic approximation of M at p is a hyperboloid.

$$2z = k_1(p)x^2 - |k_2(p)|y^2$$



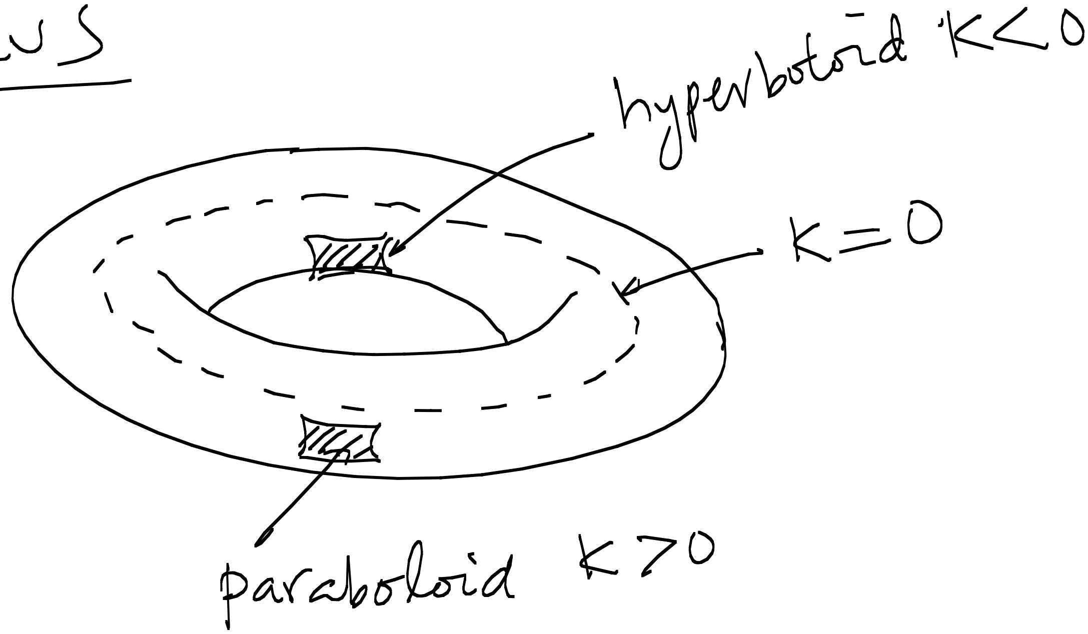
$\boxed{K(p) = 0}$ $\quad k_1(p) \neq 0 \quad k_2(p) = 0.$

$\Rightarrow$ Cylinder.



$k_1(p) = k_2(p) = 0$
$\Rightarrow$ plane

<u>TORUS</u>



<u>Def</u>. M is <u>flat</u> if $K = 0$

M is <u>minimal</u> if $H = 0$

Cylinders, planes are <u>flat</u>.

$K \leq 0$ if M is minimal.

<u>Constant Gaussian Curvature</u>: Sphere

$$k_1 = k_2 = -\frac{1}{r} \quad \therefore \quad K = \frac{1}{r^2}.$$

Smaller the sphere, larger the curvature.