Foundation of Computer Science 1

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Problem 1. Consider an information system $X = \{x_1, x_2, x_3, x_4, x_5\}$ with 5 messages $x_1, \dots, x_5$. The probabilities of transmitting the messages $x_1, \dots, x_5$ are, respectively, $p_1 = 0.5$, $p_2 = 0.25$, $p_3 = 0.125$, $p_4 = 0.0625$, and $p_5 = 0.0625$.
What is the minimal length of a word code in a binary encoding of X?

Solution. The minimal word length cannot be bigger than the average word length $L(X)$ of X, nor smaller than the entropy $E(X)$ of X. To this end, as $L(X) \geq E(X)$, the minimal word length is (lower-)bounded by $E(X)$, and it is the closest integer to $E(X)$.
We have:

$$E(X) = \sum_{i=1}^{5} p_i \text{ld}(1/p_i) = \frac{1}{2}\text{ld}(2) + \frac{1}{4}\text{ld}(4) + \frac{1}{8}\text{ld}(8) + 2 \cdot \frac{1}{16}\text{ld}(16) = \frac{15}{8} = 1.875\text{bits}$$

Hence, the minimal word length is 2 bits.

Problem 2. Let p, q and r be propositional formulas.

2.1 By computing truth tables, show that $p \wedge q \wedge (\neg p \vee \neg q)$ and $\neg p \wedge \neg q \wedge (p \vee q)$ are logically equivalent.

Solution.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q \wedge (\neg p \vee \neg q)$	$p \vee q$	$\neg p \wedge \neg q \wedge (p \vee q)$
0	0	1	1	1	0	0	0
0	1	1	0	1	0	1	0
1	0	0	1	1	0	1	0
1	1	0	0	0	0	1	0

The formulas $p \wedge q \wedge (\neg p \vee \neg q)$ and $\neg p \wedge \neg q \wedge (p \vee q)$ are logically equivalent as they have the same truth values.

2.2 Prove that $(p \implies r) \wedge (q \implies r)$ and $(p \vee q) \implies r$ are logically equivalent. Your proof should rely on using equivalent transformations rules.

Solution.

$$(p \vee q) \implies r$$
$$\overset{Implication}{\Longleftrightarrow}$$
$$\neg(p \vee q) \vee r$$
$$\overset{deMorgan}{\Longleftrightarrow}$$
$$(\neg p \wedge \neg q) \vee r$$
$$\overset{Distributivity}{\Longleftrightarrow}$$
$$(\neg p \vee r) \wedge (\neg q \vee r)$$
$$\overset{Implication}{\Longleftrightarrow}$$
$$(p \implies r) \wedge (q \implies r)$$

2.3 Consider the propositional formula

$$(p \lor q) \land (\neg p \lor r) \implies (q \lor r).$$

By using those simplification rules which you consider appropriate, show that it is a tautology.

Solution.

$$
\begin{aligned}
& (p \lor q) \land (\neg p \lor r) \implies (q \lor r) \\
\overset{Implication}{\iff} \quad & \neg\big((p \lor q) \land (\neg p \lor r)\big) \lor (q \lor r) \\
\overset{deMorgan}{\iff} \quad & \big(\neg(p \lor q) \lor \neg(\neg p \lor r)\big) \lor q \lor r \\
\overset{deMorgan}{\iff} \quad & (\neg p \land \neg q) \lor (p \land \neg r) \lor q \lor r \\
\overset{Associativity}{\iff} \quad & (\neg p \land \neg q) \lor q \lor (p \land \neg r) \lor r \\
\overset{Distributivity}{\iff} \quad & \big((\neg p \lor q) \land (\neg q \lor q)\big) \lor \big((p \lor r) \land (\neg r \lor r)\big) \\
\iff \quad & \big((\neg p \lor q) \land 1\big) \lor \big((p \lor r) \land 1\big) \\
\iff \quad & (\neg p \lor q) \lor (p \lor r) \\
\overset{Associativity}{\iff} \quad & \neg p \lor p \lor q \lor r \\
\iff \quad & 1 \lor q \lor r \\
\iff \quad & 1
\end{aligned}
$$

Hence, $(p \lor q) \land (\neg p \lor r) \implies (q \lor r)$ is always 1, and therefore it is a tautology.

2.4 Consider the propositional formula

$$(p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r)$$

Is this formula satisfiable? Is it a tautology? Is it a contradiction?

Solution. Let us first simplify $(p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r)$.

$$
\begin{aligned}
& (p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r) \\
\overset{Distributivity}{\iff} \quad & (p \lor \neg r) \land (q \lor \neg q) \\
\iff \quad & (p \lor \neg r) \land 1 \\
\iff \quad & p \lor \neg r
\end{aligned}
$$

Hence, formula $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$ is logically equivalent to $(p \vee \neg r)$. That is, $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$ is satisfiable/tautology/contradiction iff $(p \vee \neg r)$ is, respectively, satisfiable/tautology/contradiction.

Formula $(p \vee \neg r)$ is

- satisfiable, for example when $p = 1$;
- is not a contradiction (as it is satisfiable, it cannot be always 0);
- is not a tautology as it is not 1 always (for example, it is 0 when $p = 0$ and $r = 1$).

Thus, formula $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$ is satisfiable, and it is not a tautology, nor a contradiction.

Note: An alternative solution could have been to compute the truth table of $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$, and conclude from the truth table whether the formula is satisfiable/tautology/contradiction.

Problem 3. Let x denote a variable, and P, Q and R be predicate formulas such that the variable x does not occur as a free variable in Q. Assume that the domain of variables is nonempty.

3.1 Prove using equivalent transformation rules the following formula:

$$((\exists x :: P(x)) \implies Q) \implies ((\forall x :: P(x)) \implies Q)$$

Solution.

$$
\begin{aligned}
& ((\exists x :: P(x)) \implies Q) \implies ((\forall x :: P(x)) \implies Q) \\
\overset{Implication}{\iff} \quad & \neg\big((\exists x :: P(x)) \implies Q\big) \vee \big((\forall x :: P(x)) \implies Q\big) \\
\overset{Implication}{\iff} \quad & \neg\big(\neg(\exists x :: P(x)) \vee Q\big) \vee \neg(\forall x :: P(x)) \vee Q \\
\overset{deMorgan}{\iff} \quad & \big((\exists x :: P(x)) \wedge \neg Q\big) \vee (\exists x :: \neg P(x)) \vee Q \\
\overset{Associativity}{\iff} \quad & \big((\exists x :: P(x)) \wedge \neg Q\big) \vee Q \vee (\exists x :: \neg P(x)) \\
\overset{Distributivity}{\iff} \quad & \Big(\big((\exists x :: P(x)) \vee Q\big) \wedge \big(\neg Q \vee Q\big)\Big) \vee (\exists x :: \neg P(x)) \\
\iff \quad & \Big(\big((\exists x :: P(x)) \vee Q\big) \wedge 1\Big) \vee (\exists x :: \neg P(x)) \\
\iff \quad & (\exists x :: P(x)) \vee Q \vee (\exists x :: \neg P(x))
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Associativity}{\Longleftrightarrow} \\
&\qquad (\exists x :: P(x)) \vee (\exists x :: \neg P(x)) \vee Q \\
&\overset{Distributivity\ \exists\vee}{\Longleftrightarrow} \\
&\qquad \Big(\exists x :: \big(P(x) \vee \neg P(x)\big)\Big) \vee Q \\
&\Longleftrightarrow \\
&\qquad (\exists x :: 1) \vee Q \\
&\Longleftrightarrow \\
&\qquad 1 \vee Q \\
&\Longleftrightarrow \\
&\qquad 1
\end{aligned}
$$

3.2 Establish the logical equivalence:

$$\forall x :: (P(x) \implies Q) \Leftrightarrow (\exists x :: P(x)) \implies Q$$

Solution.

$$
\begin{aligned}
&\qquad \forall x :: (P(x) \implies Q) \\
&\overset{Implication}{\Longleftrightarrow} \\
&\qquad \forall x :: (\neg P(x) \vee Q) \\
&\overset{x \text{ is not a free variable in } Q}{\Longleftrightarrow} \\
&\qquad (\forall x :: \neg P(x)) \vee Q \\
&\overset{deMorgan}{\Longleftrightarrow} \\
&\qquad \neg(\exists x :: P(x)) \vee Q \\
&\overset{Implication}{\Longleftrightarrow} \\
&\qquad (\exists x :: P(x)) \implies Q
\end{aligned}
$$

3.3 Suppose that the domain of x consists of -4, -2, 4, and 8. Express the statements below without using quantifiers, instead using only negations, conjunctions and disjunctions.

(a) $\forall x :: P(x)$;

(b) $\exists x :: (\neg P(x)) \wedge \forall x :: ((x \neq 2) \implies P(x))$

Solution.

(a) $P(-4) \wedge P(-2) \wedge P(4) \wedge P(8)$;

(b) Note that $\exists x :: (\neg P(x))$ is logically equivalent to $\neg P(-4) \vee \neg P(-2) \vee \neg P(4) \vee \neg P(8)$. Further, $\forall x :: ((x \neq 2) \implies P(x))$ is logically equivalent to $\forall x :: ((x = 2) \vee P(x))$, which is logically equivalent to $P(-4) \wedge P(-2) \wedge P(4) \wedge P(8)$. [1]

Thus, $\exists x :: (\neg P(x)) \wedge \forall x :: ((x \neq 2) \implies P(x))$ is logically equivalent to

$$\big(\neg P(-4) \vee \neg P(-2) \vee \neg P(4) \vee \neg P(8)\big) \wedge P(-4) \wedge P(-2) \wedge P(4) \wedge P(8),$$

that is always 0.

[1] We have $(-4 = 2 \vee P(-4)) \wedge (-2 = 2 \vee P(-2)) \wedge (4 = 2 \vee P(4)) \wedge (8 = 2 \vee P(8))$, which is equivalent to $P(-4) \wedge P(-2) \wedge P(4) \wedge P(8)$.

3.4 What is the truth value of the formula $(\text{Anz } x :: P(x)) = 1 \implies \exists x :: P(x)$?

Solution. The truth value of the formula is 1.

Explanation:
Assume that $(\text{Anz } x :: P(x)) = 1$ holds. Thus, the number of objects x such that $P(x)$ holds is 1. Therefore, there exists an x (actually, only one x) such that $P(x)$ holds. Hence, $\exists x :: P(x)$ holds.

Note: However, $(\text{Anz } x :: P(x)) = 1 \iff \exists x :: P(x)$ does NOT hold in general! (Why?)

3.5 Determine whether $\forall x :: (P(x) \implies R(x))$ and $\forall x :: P(x) \implies \forall x :: R(x)$ are logically equivalent. Justify your answer!

Solution. They are not equivalent.

Let $P(x)$ be any predicate that is sometimes 1 and sometimes 0. Let $R(x)$ be a predicate that is always 0 (independently what the value of x is).

Note that $\forall x :: P(x)$ is 0. Then, $\forall x :: P(x) \implies \forall x :: R(x)$ is 1, but $\forall x :: (P(x) \implies R(x))$ is 0. Hence, $\forall x :: P(x) \implies \forall x :: R(x)$ and $\forall x :: (P(x) \implies R(x))$ have different truth values, therefore they are not logically equivalent.