Foundation of Computer Science 1 Take Home Exercise Set 1

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Problem 1. Information Theory

A traffic controller signals the message "STOP" with a probability of $\frac{1}{4}$, the message "PREPARE" with a probability of $\frac{1}{2}$, and the message "START" with a probability of $\frac{1}{4}$.

(a) What is the information content of signalling the message "STOP"?

(b) What is the information content of signalling the message "START"?

(c) What is the information content of signalling the message "PREPARE"?

(d) What is the information content of signalling an event?

Problem 2. Boolean Algebra (Propositional Logic)

(2.1) Construct the truth table for each of the following propositional formulas, and determine whether the formulas are tautologies, satisfiable, or contradictions.

(a) $p \wedge \neg p$ **(b)** $p \vee \neg p$ **(c)** $(p \vee \neg q) \implies q$

(d) $(p \vee q) \implies (p \wedge q)$ **(e)** $(p \implies q) \Leftrightarrow (\neg q \implies \neg p)$ **(f)** $(p \implies q) \implies q$

(g) $(p \vee q) \implies (p \text{ XOR } q)$ **(h)** $(p \text{ XOR } q) \implies (p \wedge q)$ **(i)** $p \implies \neg q$

(j) $\neg p \Leftrightarrow q$ **(k)** $(p \implies q) \vee (\neg p \implies q)$ **(l)** $(p \Leftrightarrow q) \vee (\neg p \Leftrightarrow \neg q)$

(2.2) Using equivalent transformation rules, show that the formulas below are tautologies.

(a) $(p \wedge q) \implies (p \vee q)$ **(b)** $(p \implies q) \wedge (q \implies r) \implies (p \implies r)$

(c) $(p \vee q) \wedge (\neg p \vee r) \implies (q \vee r)$ **(d)** $(p \Leftrightarrow q) \Leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$

(2.3) Using equivalent transformation rules, show that:

(a) $p \implies q$ and $\neg q \implies \neg p$ are logically equivalent;

(b) $\neg p \Leftrightarrow q$ and $p \Leftrightarrow \neg q$ are logically equivalent;

(c) $\neg(p \Leftrightarrow q)$ and $\neg p \Leftrightarrow q$ are logically equivalent.

(2.4) Find the dual of each of the below formulas:

(a) $p \wedge \neg q \wedge \neg r$ **(b)** $(p \wedge q \wedge r) \vee s$

(c) $(p \vee 0) \wedge (q \vee q)$ **(d)** $p \vee \neg q$

(e) $p \wedge (q \vee (r \wedge 1))$ **(f)** $(p \wedge \neg q) \vee (q \wedge 0)$

Problem 3. Predicate Logic

(3.1) Suppose that the domain of the variable x consists of 1, 2, 3, and 4. What is the truth value of the formulas below, where $P(x)$ is the statement " $x^2 < 10$ ".

(a) $\forall x :: P(x)$ **(b)** $\exists x : P(x)$ **(c)** $P(3)$

(3.2) Suppose that the domain of the variable x consists of -5, -3, -1, 1, 3 and 5. Express (and simplify) the statements below without using quantifiers, instead using only negations, conjunctions and disjunctions.

(a) $\exists x :: P(x)$ **(b)** $\forall x : P(x)$

(b) $\forall x :: ((x \neq 1) \implies P(x))$ **(c)** $\exists x : ((x \geq 0) \wedge P(x))$

(d) $\neg\exists x :: P(x)$ **(e)** $\neg\forall x : P(x)$

(f) $\forall x :: ((x < 0) \implies P(x))$ **(g)** $\forall x : ((x \neq 3) \implies P(x))$

(3.3) Suppose that the domain of the variable x is the set of natural numbers $\mathbb{N}$. What are the truth values of the formulas below:

(a) $\exists x :: (x^2 = 2)$ **(b)** $\forall x : (x^2 = 2)$

(c) $\exists x :: (x^2 = -1)$ **(d)** $\forall x : (x^2 \neq x)$

(3.4) Let x denote a variable, and P be a predicate formula. Assume that the domain of variables is nonempty.

Using equivalent transformation rules, prove the formulas below:

(a) $\forall x :: P(x) \implies \exists x :: P(x)$;

(b) $\neg(\exists x :: \neg P(x)) \implies \forall x :: P(x)$

Problem 4. Relations

(4.1) Consider the binary relation $\equiv_3$ over the integer numbers $\mathbb{Z}$ defined as below:

$$a \equiv_3 b \quad \text{if and only if} \quad a - b \text{ is divisible by 3.}$$

(a) Show that $\equiv_3$is an equivalence relation!

(b) Is $\equiv_3$ a total relation?

(c) Is $\equiv_3$ a partial order?

(d) Is $\equiv_3$ irreflexive?

(2.2) For each of the following relations R on the set of integer numbers $\mathbb{Z}$, determine whether the relation is reflexive, symmetric, transitive, or antisymmetric.

(a) aRb if and only if $a + b$ is odd;

(b) aRb if and only if $a + b$ is even;

(c) aRb if and only if $a = b^2$;

(d) aRb if and only if $a - b = 0$;

(2.3) Consider the binary relation $|$ over the integer numbers $\mathbb{Z}$ defined as below:

$$a|b \quad \text{if and only if} \quad a \text{ divides } b.$$

(a) Show that $|$ is a partial order;

(b) Consider the set $X = \{1, 2, 3, 6\}$. Give an upper bound for X with respect to the relation $|$. What is $lub(X)$?

(b) Consider the set $X_2 = \{2, 4, 6, 12\}$. Give an upper bound for X with respect to the relation $|$. What is $lub(X)$?

Problem 5. Program Verification

(5.1) Let x and y be program variables with values from the natural numbers $\mathbb{N}$. What is:

(a) $\text{wp}(x := x + 1,\ x > 2)$?
(b) $\text{wp}(y := y + 1,\ x > 2)$?
(c) $\text{wp}(x := x + y; y := y + 1,\ x > 2)$?
(d) $\text{wp}(y := y + x; x := x + y,\ x > 2)$?
(e) $\text{wp}(y := y + x; x := x + y,\ \mathit{True})$?
(f) $\text{wp}(\underline{\text{if}}\ x \leq 2\ \underline{\text{then}}\ x := x + y; y := y + x\ \underline{\text{else}}\ x := x - y; y := y + x,\ x = 3 \wedge y = 3)$?
(f) $\text{wp}(\underline{\text{if}}\ x \geq 2\ \underline{\text{then}}\ x := x + y; y := y + x\ \underline{\text{else}}\ x := x - y; y := y + x,\ x > 10 \wedge y > 10)$?

(5.2) Let x and y be program variables with values from the natural numbers $\mathbb{N}$. Which of the following Hoare triples are correct?

(a) $\{x = 1\}\quad \underline{\text{if}}\ x \geq 2\ \underline{\text{then}}\ x := x - 1\ \underline{\text{else}}\ x := x + 1 \quad \{x = 2\}$

(b) $\{x = 1\}\quad \underline{\text{if}}\ x \geq 2\ \underline{\text{then}}\ x := x - 1\ \underline{\text{else}}\ x := x + 1 \quad \{x = 0\}$

(c) $\{x = 1 \wedge y = 1\}\quad \underline{\text{if}}\ x < 2\ \underline{\text{then}}\ x := x + y\ \underline{\text{else}}\ x := x - y \quad \{x = 2 \wedge y = 1\}$

(d) $\{x = 1 \wedge y = 1\}\quad \underline{\text{if}}\ x < 2\ \underline{\text{then}}\ x := x + y\ \underline{\text{else}}\ x := x - y \quad \{x = 0 \wedge y = 1\}$

(d) $\{x = 2 \wedge y = 1\}\quad \underline{\text{if}}\ x \leq 2\ \underline{\text{then}}\ x := x + y; y := y + x\ \underline{\text{else}}\ x := x - y; y := y + x \quad \{x = 3 \wedge y = 4\}$

(e) $\{x = 2 \wedge y = 1\}\quad \underline{\text{if}}\ x \leq 2\ \underline{\text{then}}\ x := x + y; y := y + x\ \underline{\text{else}}\ x := x - y; y := y + x \quad \{x = 3 \wedge y = 3\}$

(5.3) Let x and y be program variables with values from the natural numbers $\mathbb{N}$. Consider the Hoare triple:

$\{x = 0 \wedge y = 0\}\quad \underline{\text{while}}\ (x < 2009)\ \underline{\text{do}}\ x := x+1; y := y+1\ \underline{\text{end while}}\ \{x = 2009 \wedge y = 2009\}$

Which of the following statements are true (and which are false)?

(a) $x = y \wedge x \leq 2009$ is an invariant;
(b) $x = y \wedge x < 2009$ is an invariant;
(c) $x = 2009$ is an invariant.

(5.3) Let x and y be program variables with values from the natural numbers $\mathbb{N}$. Consider the Hoare triple:

$\{x = 0 \wedge y = 0\}\quad \underline{\text{while}}\ (x < 10)\ \underline{\text{do}}\ x := x + 1; y := y + 1\ \underline{\text{end while}}\ \{x = 10 \wedge y = 10\}$,

annotated with the loop invariant $x \leq 10 \wedge x = y$.

(a) What is $\text{wp}(\underline{\text{while}}\ (x < 10)\ \underline{\text{do}}\ x := x + 1; y := y + 1\ \underline{\text{end while}},\ x = 10 \wedge y = 10)$?
(b) What are the verification conditions of the above given Hoare triple?

(5.4) Let x and y be program variables with values from the integer numbers $\mathbb{Z}$. Consider the Hoare triple:

$\{x = 0 \wedge y = 0\}\quad \underline{\text{while}}\ (x < 5)\ \underline{\text{do}}\ y := y + x; x := x + 1\ \underline{\text{end while}}\quad \{x = 5 \wedge y = 10\}$,

annotated with the loop invariant $(x^2 = 2 * y + x) \wedge (x \leq 5)$.

What are the verification conditions of the above given Hoare triple?